

Perfect Packings in Quasirandom Hypergraphs I

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Abstract

Let $k \geq 2$ and F be a linear k -uniform hypergraph with v vertices. We prove that if n is sufficiently large and $v|n$, then every quasirandom k -uniform hypergraph on n vertices with constant edge density and minimum degree $\Omega(n^{k-1})$ admits a perfect F -packing. The case $k = 2$ follows immediately from the blowup lemma of Komlós, Sárközy, and Szemerédi. We also prove positive results for some nonlinear F but at the same time give counterexamples for rather simple F that are close to being linear. Finally, we address the case when the density tends to zero, and prove (in analogy with the graph case) that sparse quasirandom 3-uniform hypergraphs admit a perfect matching as long as their second largest eigenvalue is sufficiently smaller than the largest eigenvalue.

1 Introduction

A k -uniform hypergraph H (k -graph for short) is a collection of k -element subsets (edges) of a vertex set $V(H)$. For a k -graph H and a subset S of vertices of size at most $k - 1$, let $d(S) = d_H(S)$ be the number of subsets of size $k - |S|$ that when added to S form an edge of H . The *minimum degree* of H , written $\delta(H)$, is the minimum of $d(\{s\})$ over all vertices s . The *minimum ℓ -degree* of H , written $\delta_\ell(H)$, is the minimum of $d(S)$ taken over all ℓ -sets of vertices. The *minimum codegree* of H is the minimum $(k - 1)$ -degree. Let K_t^k be the complete k -graph on t vertices.

Let G and F be k -graphs. We say that G has a *perfect F -packing* if the vertex set of G can be partitioned into copies of F . An important result of Hajnal and Szemerédi [9] states that if r divides n and the minimum degree of an n -vertex graph G is at least $(1 - 1/r)n$, then G has a perfect K_r -packing. Later Alon and Yuster [2] conjectured that a similar result holds for any graph F instead of just cliques, with the minimum degree of G depending on

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the chromatic number of F . This was proved by Komlós-Sárközy-Szemerédi [18] by using the Regularity Lemma and Blow-up Lemma. Later, Kühn and Osthus [20] found the minimum degree threshold for perfect F -packings up to a constant; the threshold either comes from the chromatic number of F or the so-called critical chromatic number of F .

In the past decade there has been substantial interest in extending this result to k -graphs. Nevertheless, the simplest case of determining the minimum codegree threshold that guarantees a perfect matching was settled only recently by Rödl-Ruciński-Szemerédi [30]. Since then, there are a few results for codegree thresholds for packing other small 3-graphs [5, 13, 19, 26, 27, 29, 33, 34]. For ℓ -degrees with $\ell < k/2$ (in particular the minimum degree), much less is known. After work by many researchers [10, 14, 15, 22, 21, 28], still only the degree threshold for K_3^3 -packings and K_4^3 -packings are known ($\frac{5}{9}$ and $\frac{37}{64}$ respectively). For $m \geq 5$ and $k \geq 4$ the packing degree threshold for K_m^k is open ([21] contains the current best bounds).

A key ingredient in the proofs of most of the above results are specially designed random-like properties of k -graphs that imply the existence of perfect F -packings. There is a rather well-defined notion of quasirandomness for graphs that originated in early work of Thomason [31, 32] and Chung-Graham-Wilson [3] which naturally generalizes to k -graphs. Our main focus in this paper is on understanding when perfect F -packings exist in quasirandom hypergraphs. The basic property that defines quasirandomness is uniform edge-distribution, and this extends naturally to hypergraphs. Let $v(H) = |V(H)|$.

Definition. Let $k \geq 2$, let $0 < \mu, p < 1$, and let H be a k -graph. We say that H is (p, μ) -dense if for all $X_1, \dots, X_k \subseteq H$,

$$e(X_1, \dots, X_k) \geq p|X_1| \cdots |X_k| - \mu n^k,$$

where $e(X_1, \dots, X_k)$ is the number of $(x_1, \dots, x_k) \in X_1 \times \cdots \times X_k$ such that $\{x_1, \dots, x_k\} \in H$ (note that if the X_i s overlap an edge might be counted more than once). Say that H is an (n, p, μ) k -graph if H has n vertices and is (p, μ) -dense. Finally, if $0 < \alpha < 1$, then an (n, p, μ) k -graph is an (n, p, μ, α) k -graph if its minimum degree is at least $\alpha \binom{n}{k-1}$.

The F -packing problem for quasirandom graphs with constant density has been solved implicitly by Komlós-Sárközy-Szemerédi [17] in the course of developing the Blow-up Lemma.

Theorem 1. (Komlós-Sárközy-Szemerédi [17]) Let $0 < \alpha, p < 1$ be fixed and let F be any graph. There exists an n_0 and $\mu > 0$ such that if H is any (n, p, μ, α) 2-graph where $n \geq n_0$, $v(F)|n$ then H has a perfect F -packing.

Note that the condition on minimum degree is required, since if the condition “ $\delta(H) \geq \alpha n$ ” in Theorem 1 is replaced by “ $\delta(H) \geq f(n)$ ” for any choice of $f(n)$ with $f(n) = o(n)$, then there exists the following counterexample. Take the disjoint union of the random graph $G(n, p)$ and a clique of size either $\lceil f(n) \rceil + 1$ or $\lceil f(n) \rceil + 2$ depending on which is odd. The minimum degree is at least $f(n)$, there is no perfect matching, and the graph is still (p, μ) -dense. Because of the use of the regularity lemma, the constant n_0 in Theorem 1 is an exponential tower in μ^{-1} . We extend Theorem 1 to a variety of k -graphs. In the process,

we also reduce the size of n_0 for all 2-graphs. A basic problem in this area that naturally emerges is the following.

Problem 2. *For which k -graphs F does the following hold: for all $0 < p, \alpha < 1$, there is some n_0 and μ so that if H is an (n, p, μ, α) k -graph with $n \geq n_0$ and $v(F)|n$, then H has a perfect F -packing.*

Unlike the graph case, most F will not satisfy Problem 2. Indeed, Rödl observed that for all $\mu > 0$ and there is an n_0 such that for $n \geq n_0$, an old construction of Erdős and Hajnal [6] produces an n -vertex 3-graph which is $(\frac{1}{4}, \mu)$ -dense and has no copy of K_4^3 . In a forthcoming paper we will show that a stronger notion of quasirandomness suffices to perfectly pack all F .

A hypergraph is *linear* if every two edges share at most one vertex. For a k -graph H , Kohayakawa-Nagle-Rödl-Schacht [16] recently proved an equivalence between $(|H|/\binom{n}{k}, \mu)$ -dense and the fact that for each linear k -graph F , the number of labeled copies of F in H is the same as in the random graph with the same density. This leads naturally to the question of whether Problem 2 has a positive answer for linear k -graphs, and our first result shows that this is the case.

Theorem 3. *Let $k \geq 2$, $0 < \alpha, p < 1$, and let F be a linear k -graph. There exists an n_0 and $\mu > 0$ such that if H is an (n, p, μ, α) k -graph where $n \geq n_0$ and $v(F)|n$, then H has a perfect F -packing.*

We restrict our attention only to 3-graphs now although the concepts extend naturally to larger k . Define a 3-graph to be $(2+1)$ -linear if its edges can be ordered as e_1, \dots, e_q such that each e_i has a partition $s_i \cup t_i$ with $|s_i| = 2, |t_i| = 1$ and for every $j < i$ we have $e_j \cap e_i \subseteq s_i$ or $e_j \cap e_i \subseteq t_i$. In words, every edge before e_i intersects e_i in a subset of s_i or of t_i . Clearly every linear 3-graph is $(2+1)$ -linear, but the converse is false. Keevash's [11] recent proof of the existence of designs and our recent work on quasirandom properties of hypergraphs [23, 24, 25] use a quasirandom property distinct from (p, μ) -dense that Keevash calls *typical* and we call $(2+1)$ -quasirandom (although the properties are essentially equivalent). These properties imply that the count of all $(2+1)$ -linear 3-graphs in a typical 3-graph is the same as in the random 3-graph (see [23, 24]).

Thus a natural direction in which to extend Theorem 3 is to the family of $(2+1)$ -linear 3-graphs and we begin this investigation with some of the smallest such 3-graphs. A *cherry* is the 3-graph comprising two edges that share precisely two vertices - this is the “simplest” non-linear hypergraph. A more complicated $(2+1)$ -linear 3-graph is $C_4(2+1)$ which has vertex set $\{1, 2, 3, 4, a, b\}$ and edge set $\{12a, 12b, 34a, 34b\}$. The importance of $C_4(2+1)$ lies in the fact that $C_4(2+1)$ is forcing for the class of all $(2+1)$ -linear 3-graphs. This means that if F is a $(2+1)$ -linear 3-graph and $p, \epsilon > 0$ are fixed, there is n_0 and $\delta > 0$ so that if $n \geq n_0$ and H is an n -vertex 3-graph with $p\binom{n}{3}$ edges and $(1 \pm \delta)p^4 n^6$ labeled copies of $C_4(2+1)$, then the number of labeled copies of F in H is $(1 \pm \epsilon)p^{|F|}n^{v(F)}$ (see [23, 24]).

Theorem 4. *Let $0 < \alpha, p < 1$. There exists an n_0 and $\mu > 0$ such that if H is an (n, p, μ, α) 3-graph where $n \geq n_0$, then H has a perfect cherry-packing if $4|n$ and a perfect $C_4(2+1)$ packing if $6|n$.*

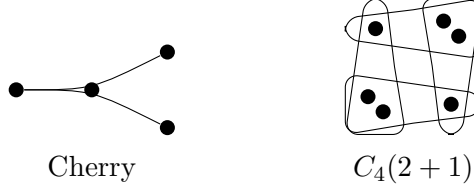


Figure 1: Two 3-graphs

One might speculate that Theorem 4 can be extended to the collection of all $(2+1)$ -linear F or to the collection of all 3-partite F . However, our next result shows that this is not the case and that solving Problem 2 will be a difficult project. If x is a vertex in a 3-graph H , the *link* of x is the graph with vertex set $V(H) \setminus \{x\}$ and edges those pairs who form an edge with x .

Theorem 5. *Let F be any 3-graph with an even number of vertices such that there exists a partition of the vertices of F into pairs such that each pair has a common edge in their links. Then for any $\mu > 0$, there exists an n_0 such that for all $n \geq n_0$, there exists a 3-graph H such that*

- $|H| = \frac{1}{8} \binom{n}{3} \pm \mu n^3$,
- H is $(\frac{1}{8}, \mu)$ -dense,
- $\delta(H) \geq (\frac{1}{8} - \mu) \binom{n}{2}$,
- H has no perfect F -packing.

Two examples of 3-graphs F that satisfy the conditions of Theorem 5 are the complete 3-partite 3-graph $K_{2,2,2}$ with parts of size two and the following $(2+1)$ -linear hypergraph. A *cherry 4-cycle* is the $(2+1)$ -linear 3-graph with edge set $\{123, 124, 345, 346, 567, 568, 781, 782\}$.

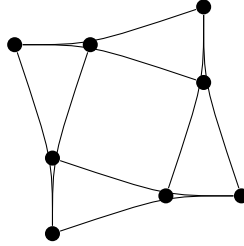


Figure 2: cherry 4-cycle

It is straightforward to see that Theorem 5 applies to the cherry 4-cycle. Therefore one cannot hope that Theorem 3 holds for all $(2+1)$ -linear or 3-partite F .

Our final result considers the situation when the density is not fixed and goes to zero. Here the notion of quasirandom is measured by spectral gap. It is a folklore result that large

spectral gap guarantees a perfect matching in graphs. For hypergraphs, there are several definitions of eigenvalues. We will use the definitions that originated in the work of Friedman and Wigderson [7, 8] for regular hypergraphs. The definition for all hypergraphs can be found in [23, Section 3] where we specialize to $\pi = 1 + \dots + 1$. That is, let $\lambda_1(H) = \lambda_{1,1+\dots+1}(H)$ and let $\lambda_2(H) = \lambda_{2,1+\dots+1}(H)$, where both $\lambda_{1,1+\dots+1}(H)$ and $\lambda_{2,1+\dots+1}(H)$ are as defined in Section 3 of [23]. The only result about eigenvalues that we will require is Proposition 24, which is usually called the Expander Mixing Lemma [23, Theorem 4] (see also [7, 8]).

Theorem 6. *For every $\alpha > 0$, there exists n_0 and $\gamma > 0$ depending only on α such that the following holds. Let H be an n -vertex 3-graph where $3|n$ and $n \geq n_0$. Let $p = 6|H|/n^3$ and assume that $\delta_2(H) \geq \alpha pn$ and*

$$\lambda_2(H) \leq \gamma p^{16} n^{3/2}.$$

Then H contains a perfect matching.

Let $\Delta_2(H)$ be the *maximum codegree* of a 3-graph H , i.e. the maximum of $d(S)$ over all 2-sets $S \subseteq V(H)$. If $\Delta_2(H) \leq cpn$ then $\lambda_1(H) \leq c'pn^{3/2}$ where c' is a constant depending only on c . This implies the following corollary.

Corollary 7. *For every $\alpha > 0$, there exists n_0 and $\gamma > 0$ depending only on α such that the following holds. Let H be an n -vertex 3-graph where $3|n$ and $n \geq n_0$. Let $p = |H|/\binom{n}{3}$ and assume that $\delta_2(H) \geq \alpha pn$, $\Delta_2(H) \leq \frac{1}{\alpha}pn$, and*

$$\lambda_2(H) \leq \gamma p^{15} \lambda_1(H).$$

Then H contains a perfect matching.

The third largest eigenvalue of a graph is closely related to its matching number (see e.g. [4]), but currently we do not know the “correct” definition of λ_3 for hypergraphs. It would be interesting to discover a definition of λ_3 for k -graphs which extends the graph definition and for which a bound on λ_3 forces a perfect matching.

The remainder of this paper is organized as follows. In Section 2 we will develop the tools necessary for our proofs, including extensions of the absorbing technique and various embedding lemmas. Then in Section 3 we will use these to prove Theorem 3 (Section 3.3) and Theorem 4 (Sections 3.1 and 3.2). Section 4 contains the construction proving Theorem 5 and Section 5 has the proof of the sparse case, Theorem 6.

2 Tools

In this section, we state and prove several lemmas and propositions that we will need; our main tool is the absorbing technique of Rödl-Ruciński-Szemerédi [30].

Definition. *Let F and H be k -graphs and let $A, B \subseteq V(H)$. We say that A F -absorbs B or that A is an F -absorbing set for B if both $H[A]$ and $H[A \cup B]$ have perfect F -packings. When F is a single edge, we say that A edge-absorbs B .*

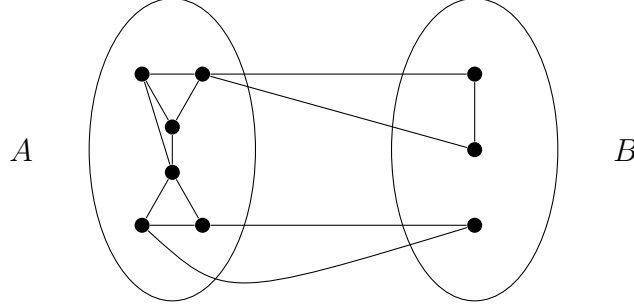


Figure 3: A K_3 -absorbs B

Definition. Let F and H be k -graphs, $\epsilon > 0$, a and b be multiples of $v(F)$, $\mathcal{A} \subseteq \binom{V(H)}{a}$, and $\mathcal{B} \subseteq \binom{V(H)}{b}$. We say that H is $(\mathcal{A}, \mathcal{B}, \epsilon, F)$ -rich if for all $B \in \mathcal{B}$ there are at least ϵn^a sets in \mathcal{A} which F -absorb B . If $\mathcal{A} = \binom{V(H)}{a}$, we abbreviate this to $(a, \mathcal{B}, \epsilon, F)$ -rich and if both $\mathcal{A} = \binom{V(H)}{a}$ and $\mathcal{B} = \binom{V(H)}{b}$, we abbreviate this to (a, b, ϵ, F) -rich.

The following proposition is one of the main results of this section; the proof appears in Section 2.3.

Proposition 8. Fix $0 < p < 1$, let F be a k -graph such that F is either linear or k -partite, and let a and b be multiples of $v(F)$. For any $\epsilon > 0$, there exists an n_0 and $\mu > 0$ such that the following holds. If H is an (a, b, ϵ, F) -rich, (n, p, μ) k -graph where $v(F) | n$, then H has a perfect F -packing.

We will actually need a slight extension of Proposition 8 for some of our results and this requires an additional definition.

Definition. Let $\zeta > 0$, t be any integer, H be a 3-graph, and $B \subseteq V(H)$ with $|B| = 2t$. We say that B is ζ -separable if there exists a partition of B into B_1, \dots, B_t such that for all i $|B_i| = 2$ and $d_H(B_i) \geq \zeta n$. Set

$$\mathcal{B}_{\zeta, b}(H) := \left\{ B \in \binom{V(H)}{b} : B \text{ is } \zeta\text{-separable} \right\}.$$

If H is obvious from context, we will denote this by $\mathcal{B}_{\zeta, b}$.

The second main result proved in this section is that the property (a, b, ϵ, F) -rich can be replaced by $(a, \mathcal{B}_{\zeta, b}, \epsilon, F)$ -rich in Proposition 8; the proof is in Section 2.4.

Proposition 9. Fix $0 < p, \alpha < 1$ and let $\zeta = \min\{\frac{p}{4}, \frac{\alpha}{4}\}$. Let F be a 3-graph such that F is either linear or k -partite, let $v(F) | a$, and let $v(F) | b$ where in addition b is even. For any $\epsilon > 0$, there exists an n_0 and $\mu > 0$ such that the following holds. If H is an $(a, \mathcal{B}_{\zeta, b}, \epsilon, F)$ -rich (n, p, μ, α) 3-graph where $v(F) | n$, then H has a perfect F -packing.

Note that if b is even, H is a 3-graph, and $\delta(H) \geq \alpha \binom{n}{2}$, then Proposition 9 implies Proposition 8. The proofs of Propositions 8 and 9 use the absorbing technique of Rödl-Ruciński-Szemerédi [30]. The two key ingredients are the Absorbing Lemma (Lemma 10) and the Embedding Lemmas (Lemma 11 for linear and Lemma 13 for k -partite). The remainder of this section contains the statements and proofs of these lemmas plus the proofs of both propositions.

2.1 Absorbing Sets

Rödl-Ruciński-Szemerédi [30, Fact 2.3] have a slightly different definition of edge-absorbing where B has size $k + 1$ and one vertex of A is left out of the perfect matching, but the main idea transfers to our setting in a straightforward way as follows. If H is a k -graph, $A \subseteq V(H)$, and $\mathcal{A} \subseteq 2^{V(H)}$, then we say that A *partitions into sets from \mathcal{A}* if there exists a partition $A = A_1 \dot{\cup} \dots \dot{\cup} A_t$ such that $A_i \in \mathcal{A}$ for all i .

Lemma 10. (*Absorbing Lemma*) *Let F be a k -graph, $\epsilon > 0$, and a and b be multiples of $v(F)$. There exists an n_0 and $\omega > 0$ such that for all n -vertex k -graphs H with $n \geq n_0$, the following holds. If H is $(\mathcal{A}, \mathcal{B}, \epsilon, F)$ -rich for some $\mathcal{A} \subseteq \binom{V(H)}{a}$ and $\mathcal{B} \subseteq \binom{V(H)}{b}$, then there exists an $A \subseteq V(H)$ such that A partitions into sets from \mathcal{A} and A F -absorbs all sets C satisfying the following conditions: $C \subseteq V(H) \setminus A$, $|C| \leq \omega n$, and C partitions into sets from \mathcal{B} .*

Using the idea of Rödl-Ruciński-Szemerédi [30], Treglown and Zhao [33, Lemma 5.2] proved the above lemma for F a single edge, $a = 2k$, $b = k$, $\mathcal{A} = \binom{V(H)}{a}$ and $\mathcal{B} = \binom{V(H)}{b}$. For the sparse case (Theorem 6) we require a stronger version of Lemma 10 and so a proof of Lemma 10 appears in Section 5 (as a corollary of Lemma 23).

2.2 Embedding Lemmas and Almost Perfect Packings

This section contains embedding lemmas for linear and k -partite k -graphs and a simple corollary of these lemmas which produces a perfect F -packing covering almost all of the vertices.

Definition. *Let F and H be k -graphs with $V(F) = \{w_1, \dots, w_f\}$. A labeled copy of F in H is an edge-preserving injection from $V(F)$ to $V(H)$. A degenerate labeled copy of F in H is an edge-preserving map from $V(F)$ to $V(H)$ that is not an injection. Let $1 \leq m \leq f$ and let $Z_1, \dots, Z_m \subseteq V(H)$. Set $\text{inj}[F \rightarrow H; w_1 \rightarrow Z_1, \dots, w_m \rightarrow Z_m]$ to be the number of edge-preserving injections $\psi : V(F) \rightarrow V(H)$ such that $\psi(w_i) \in Z_i$ for all $1 \leq i \leq m$. In other words, $\text{inj}[F \rightarrow H; w_1 \rightarrow Z_1, \dots, w_m \rightarrow Z_m]$ is the number of labeled copies of F in H where w_i is mapped into Z_i for all $1 \leq i \leq m$. If $Z_i = \{z_i\}$, we abbreviate $w_i \rightarrow \{z_i\}$ as $w_i \rightarrow z_i$.*

Lemma 11. *Let $0 < p, \alpha < 1$ and let F be a linear k -graph where $0 \leq m \leq v(F)$ and $V(F) = \{s_1, \dots, s_m, t_{m+1}, \dots, t_f\}$ such that there does not exist $E \in F$ with $|E \cap \{s_1, \dots, s_m\}| > 1$*

and there do not exist $E_1, E_2 \in F$ with $|E_1 \cap \{s_1, \dots, s_m\}| = 1$, $|E_2 \cap \{s_1, \dots, s_m\}| = 1$, and $E_1 \cap E_2 \cap \{t_{m+1}, \dots, t_f\} \neq \emptyset$.

For every $\gamma > 0$, there exists an n_0 and $\mu > 0$ such that the following holds. Let H be an n -vertex k -graph with $n \geq n_0$ and let $y_1, \dots, y_m \in V(H)$, $Z_{m+1} \subseteq V(H), \dots, Z_f \subseteq V(H)$. Assume that for every $\{s_i, t_{j_2}, \dots, t_{j_k}\} \in F$

$$\left| \left\{ (z_{j_2}, \dots, z_{j_k}) \in Z_{j_2} \times \dots \times Z_{j_k} : \{y_i, z_{j_2}, \dots, z_{j_k}\} \in H \right\} \right| \geq \alpha |Z_{j_2}| \dots |Z_{j_k}| \quad (1)$$

and for every $\{t_{i_1}, \dots, t_{i_k}\} \in F$ and every $Z'_{i_1} \subseteq Z_{i_1}, \dots, Z'_{i_k} \subseteq Z_{i_k}$,

$$e(Z'_{i_1}, \dots, Z'_{i_k}) \geq p |Z_{i_1}| \dots |Z_{i_k}| - \mu n^k. \quad (2)$$

Then

$$\begin{aligned} \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow Z_{m+1}, \dots, t_f \rightarrow Z_f] \\ \geq \alpha^{d_F(s_1)} \dots \alpha^{d_F(s_m)} p^{|F| - \sum d_F(s_i)} |Z_{m+1}| \dots |Z_f| - \gamma n^{f-m}. \end{aligned}$$

Proof. Kohayakawa, Nagle, Rödl, and Schacht [16] proved this when $Z_i = V(H)$ for all i , without the distinguished vertices s_1, \dots, s_m , and under a stronger condition on H , but it is straightforward to extend their proof to our setup as follows. The lemma is proved by induction on number of edges of F which do not contain any vertex from among s_1, \dots, s_m . Let $\mu = (1 - p)\gamma$.

First, if every edge of F contains some s_i then F is a vertex disjoint union of stars with centers s_1, \dots, s_m plus some isolated vertices. Therefore, we can form a copy of F of the type we are trying to count by picking an edge of H containing y_i (of the right type) for each edge of F . More precisely, using (1), the fact that all edges of F which use some s_1, \dots, s_m (so all edges of F) do not share any vertices from among t_{m+1}, \dots, t_f , and the fact that F is linear, the number of labeled copies of F with $s_i \rightarrow y_i$ and $t_j \rightarrow Z_j$ is at least

$$\alpha^{|F|} |Z_{m+1}| \dots |Z_f| = \alpha^{\sum d_F(s_i)} p^0 |Z_{m+1}| \dots |Z_f|.$$

The proof of the base case is complete.

Now assume F has at least one edge E which does not contain any s_i , with vertices labeled so that $E = \{t_{m+1}, \dots, t_{m+k}\}$. Let F_* be the hypergraph formed by deleting all vertices of E from F and notice that $s_i \in V(F_*)$ for all i . Let F_- be the hypergraph formed by removing the edge E from F but keeping the same vertex set. Let Q_* be an injective edge-preserving map $Q_* : V(F_*) \rightarrow V(H)$ where $Q_*(s_i) = y_i$ for $1 \leq i \leq m$ and $Q_*(t_j) \in Z_j$ for $m+1 \leq j \leq f$. For $m+1 \leq j \leq m+k$, define $S_j(Q_*) \subseteq Z_j$ as follows. For each $z \in Z_j$, add z to $S_j(Q_*)$ if $z \notin \text{Im}(Q_*)$ and there exists an edge-preserving injection $V(F_*) \cup \{t_j\} \rightarrow \text{Im}(Q_*) \cup \{z\}$ which when restricted to $V(F_*)$ matches the map Q_* . More informally, $S_j(Q_*)$ consists of all vertices which can be used to extend Q_* to embed a labeled copy of $F_* \cup \{t_j\}$.

By definition, every edge counted by $e(S_{m+1}(Q_*), \dots, S_{m+k}(Q_*))$ creates a labeled copy of F . Also, every ordered tuple from $S_{m+1}(Q_*) \times \dots \times S_{m+k}(Q_*)$ creates a labeled copy of

F_- . More precisely,

$$\begin{aligned}
& \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow Z_{m+1}, \dots, t_f \rightarrow Z_f] \\
&= \sum_{Q_*} e(S_{m+1}(Q_*), \dots, S_{m+k}(Q_*)) \\
& \text{inj}[F_- \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow Z_{m+1}, \dots, t_f \rightarrow Z_f] \\
&= \sum_{Q_*} |S_{m+1}(Q_*)| \cdots |S_{m+k}(Q_*)|. \tag{3}
\end{aligned}$$

For each j , $S_j(Q_*) \subseteq Z_j$ so that (2) implies that

$$\begin{aligned}
& \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow Z_{m+1}, \dots, t_f \rightarrow Z_f] \\
& \geq \sum_{Q_*} (p|S_{m+1}(Q_*)| \cdots |S_{m+k}(Q_*)| - \mu n^k) \\
& \geq p \sum_{Q_*} |S_{m+1}(Q_*)| \cdots |S_{m+k}(Q_*)| - \mu n^{f-m}, \tag{4}
\end{aligned}$$

where the last inequality is because there are at most n^{f-m-k} maps Q_* , since F_* has $f-k$ vertices and $s_i \in V(F_*)$ must map to y_i . Combining (3) and (4) and then applying induction,

$$\begin{aligned}
& \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow Z_{m+1}, \dots, t_f \rightarrow Z_f] \\
& \geq p \text{inj}[F_- \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow Z_{m+1}, \dots, t_f \rightarrow Z_f] - \mu n^{f-m} \\
& \geq p (\alpha^{\sum d(s_i)} p^{|F|-1-\sum d(s_i)} |Z_{m+1}| \cdots |Z_f| - \gamma n^{f-m}) - \mu n^{f-m}.
\end{aligned}$$

Since $\mu = (1-p)\gamma$, the proof is complete. \square

Corollary 12. *Let $0 < p < 1$ and let F be a linear k -graph with $V(F) = \{t_1, \dots, t_f\}$. For every $\gamma > 0$, there exists an n_0 and $\mu > 0$ such that the following holds. Let H be an (n, p, μ) k -graph and let $Z_1, \dots, Z_f \subseteq V(H)$. Then*

$$\text{inj}[F \rightarrow H; t_1 \rightarrow Z_1, \dots, t_f \rightarrow Z_f] \geq p^{|F|} |Z_1| \cdots |Z_f| - \gamma n^f.$$

Proof. Apply Lemma 11 with $m = 0$. Since H is (p, μ) -dense, (2) holds. Also, (1) is vacuous since $m = 0$. \square

Lemma 13. *Let $0 < p < 1$ and let K_{t_1, \dots, t_k} be the complete k -partite, k -graph with part sizes t_1, \dots, t_k and parts labeled by T_1, \dots, T_k . For every $0 < \mu < \frac{p}{2}$, there exists n_0 and $0 < \xi < 1$ such that the following holds. Let H be an (n, p, μ) k -graph with $n \geq n_0$. Then for any $X_1, \dots, X_k \subseteq V(H)$ with $|X_j| \geq (2\mu/p)^{1/k} n$ for all j , the number of labeled copies of K_{t_1, \dots, t_k} in H with $T_i \subseteq X_i$ for all i is at least $\xi \prod |X_i|^{t_i}$.*

Proof. Let H' be the k -graph on $\sum |X_i|$ vertices with vertex set $Y_1 \dot{\cup} \cdots \dot{\cup} Y_t$ where the sets Y_i are disjoint and $Y_i \cong X_i$ for all i . Note that because the sets X_i might overlap, a vertex

of H might appear more than once in H' . Make $y_1 \in Y_1, \dots, y_k \in Y_k$ a hyperedge of H' if y_1, \dots, y_k are distinct vertices of H and $\{y_1, \dots, y_k\} \in H$. Let $t = \sum t_i$. Since H is (p, μ) -dense,

$$e(H') = e_H(X_1, \dots, X_k) \geq p \prod_i |X_i| - \mu n^k \geq p \left(\frac{2\mu}{p} \right) n^k - \mu n^k = \mu n^k \geq \frac{\mu}{k^k} v(H')^k.$$

Therefore, by supersaturation (see [12, Theorems 2.1 and 2.2]), there exists an n'_0 and $\xi' > 0$ such that if $v(H') \geq n'_0$ then H' contains at least $\xi' v(H')^t$ labeled copies of K_{t_1, \dots, t_k} . Each of these labeled copies of K_{t_1, \dots, t_k} in H' produces a possibly degenerate labeled copy of K_{t_1, \dots, t_k} in H where $T_i \subseteq X_i$ for all i . Pick $\xi = \frac{1}{2}\xi'$, $n_0 \geq n'_0(p/2\mu)^{1/k}$, and $n_0 \geq \frac{1}{\xi}(p/2\mu)^{t/k}$.

Now assume that $n \geq n_0$. This implies that $v(H') \geq |X_1| \geq (2\mu/p)^{1/k} n \geq n'_0$ so that there are at least $\xi' v(H')^t$ labeled copies of K_{t_1, \dots, t_k} in H' . Therefore, the number of possibly degenerate labeled copies of K_{t_1, \dots, t_k} in H with $T_i \subseteq X_i$ for all i is at least

$$\xi' v(H')^t = \xi' \prod_i v(H')^{t_i} \geq \xi' \prod_i |X_i|^{t_i} = 2\xi \prod_i |X_i|^{t_i}. \quad (5)$$

Since there are at most n^{t-1} degenerate labeled copies, by the choice of n_0 and since $|X_i| \geq (2\mu/p)^{1/k} n$ for all i , the number of degenerate labeled copies is at most

$$n^{t-1} = \frac{1}{n} \left(\frac{p}{2\mu} \right)^{t/k} \prod_i \left[\left(\frac{2\mu}{p} \right)^{1/k} n \right]^{t_i} \leq \frac{1}{n} \left(\frac{p}{2\mu} \right)^{t/k} \prod_i |X_i|^{t_i} \leq \xi \prod_i |X_i|^{t_i}. \quad (6)$$

Combining (5) with (6) shows that there are at least $\xi \prod_i |X_i|^{t_i}$ labeled copies of K_{t_1, \dots, t_k} with $T_i \subseteq X_i$ for all i , completing the proof. \square

With these lemmas in hand, we can prove that if H is (p, μ) -dense and F is linear or k -partite, then H has an F -packing covering almost all the vertices of H .

Lemma 14. (*Almost Perfect Packing Lemma*) Fix $0 < p < 1$ and a k -graph F with f vertices such that F is either linear or k -partite. Let $v(F) \mid b$. For any $0 < \omega < 1$, there exists n_0 and $\mu > 0$ such that the following holds. Let H be an (n, p, μ) k -graph with $n \geq n_0$ and $f \mid n$. Then there exists $C \subseteq V(H)$ such that $|C| \leq \omega n$, $b \mid |C|$, and $H[C]$ has a perfect F -packing.

Proof. First, select n_0 large enough and μ small enough so that any vertex set C of size $\lceil \frac{\omega}{2} \rceil$ contains a copy of F . To see that this is possible, there are two cases to consider.

If F is linear, let $\gamma = \frac{1}{2} p^{|F|} \left(\frac{\omega}{2} \right)^f$ and select n_0 and $\mu > 0$ according to Corollary 12. Now if $C \subseteq V(H)$ with $|C| \geq \frac{\omega}{2} n$, then Corollary 12 implies there are at least $p^{|F|} |C|^f - \gamma n^f \geq p^{|F|} \left(\frac{\omega}{2} \right)^f n^f - \gamma n^f = \gamma n^f > 0$ copies of F inside C .

If F is k -partite, then Lemma 13 is used in a similar way as follows. Let $\mu = \frac{p}{2} \left(\frac{\omega}{2} \right)^k$ and select n_0 and ξ according to Lemma 13. Now by the choice of μ , if $|C| \geq \frac{\omega}{2} n$ then $|C| \geq (2\mu/p)^{1/k} n$ so that by Lemma 13, C contains at least $\xi \left(\frac{\omega}{2} \right)^f n^f > 0$ copies of F .

Now let F_1, \dots, F_t be a greedily constructed F -packing. That is, F_1, \dots, F_t are disjoint copies of F and $C := V(H) \setminus V(F_1) \setminus \dots \setminus V(F_t)$ has no copy of F . By the previous two paragraphs, $|C| \leq \frac{\omega}{2}n$. Since $f|n$ and $H[\bar{C}]$ has a perfect F -packing, $f||C|$. Thus we can let $y \equiv -\frac{|C|}{f} \pmod{b}$ with $0 \leq y < b$ and take y of the copies of F in the F -packing of $H[\bar{C}]$ and add their vertices into C so that $b||C|$. \square

2.3 Proof of Proposition 8

Proof of Proposition 8. First, select $\omega > 0$ according to Lemma 10 and $\mu_1 > 0$ according to Lemma 14. Also, make n_0 large enough so that both Lemma 10 and 14 can be applied. Let $\mu = \mu_1 \omega^k$. All the parameters have now been chosen.

By Lemma 10, there exists a set $A \subseteq V(H)$ such that A F -absorbs C for all $C \subseteq V(H) \setminus A$ with $|C| \leq \omega n$ and $b \mid |C|$. If $|A| \geq (1 - \omega)n$, then A F -absorbs $V(H) \setminus A$ so that H has a perfect F -packing. Thus $|A| \leq (1 - \omega)n$. Next, let $H' := H[\bar{A}]$ and notice that H' is (p, μ_1) -dense since $v(H') \geq \omega n$ and

$$\mu n^k \leq \frac{\mu}{\omega^k} v(H')^k = \mu_1 v(H')^k.$$

Therefore, by Lemma 14, there exists a vertex set $C \subseteq V(H') = V(H) \setminus A$ such that $|C| \leq \omega n$, $|C|$ is a multiple of b , and $H'[\bar{C}]$ has a perfect F -packing. Now Lemma 10 implies that A F -absorbs C . The perfect F -packing of $A \cup C$ and the perfect F -packing of $H'[\bar{C}]$ produces a perfect F -packing of H . \square

2.4 Proof of Proposition 9

This section contains the proof of Proposition 9, but first we need an extension of Lemma 14 that produces a perfect F -packing covering almost all the vertices where in addition the unsaturated vertices are ζ -separable. To do so, we need a well-known probability lemma.

Lemma 15. (Chernoff Bound) *Let $0 < p < 1$, let X_1, \dots, X_n be mutually independent indicator random variables with $\mathbb{P}[X_i = 1] = p$ for all i , and let $X = \sum X_i$. Then for all $a > 0$,*

$$\mathbb{P}[|X - \mathbb{E}[X]| > a] \leq 2e^{-a^2/2n}.$$

Lemma 16. *Fix $p, \alpha \in (0, 1)$, $\zeta = \min\{\frac{p}{4}, \frac{\alpha}{4}\}$, and a 3-graph F such that either F is linear or F is 3-partite. Let $v(F)|b$ where in addition b is even. For any $0 < \omega < 1$, there exists n_0 and $\mu > 0$ such that the following holds. Let H be an (n, p, μ, α) 3-graph with $n \geq n_0$ and $v(F)|n$. Then there exists a set $C \subseteq V(H)$ such that $|C| \leq \omega n$, C partitions into sets of $\mathcal{B}_{\zeta, b}$, and $H[\bar{C}]$ has a perfect F -packing.*

Proof. Use Lemma 14 to select n_0 and $\mu_1 > 0$ to produce an F -packing F_1, \dots, F_t where $W := V(H) \setminus V(F_1) \setminus \dots \setminus V(F_t)$ is such that $|W| \leq \frac{\omega\alpha}{4}n$. Let $f = v(F)$ and let

$$\phi = \min \left\{ \frac{\omega}{8}, \frac{\alpha}{4}, \frac{p}{4} \right\},$$

$$\mu = \min \left\{ \frac{p}{2} \left(\frac{\alpha\phi}{16f} \right)^2, \mu_1 \right\}.$$

First, form a vertex set C' by starting with W and for each $1 \leq i \leq t$, add $V(F_i)$ to C' with probability ϕ independently. After this, take $\frac{b}{f} - \frac{|C'|}{f} \pmod{\frac{b}{f}}$ of the unselected copies of F and add their vertices into C' to form the vertex set C .

By construction, $H[\bar{C}]$ has a perfect F -packing (the copies of F which were not selected) and $b||C|$. Since b is even, $|C|$ is also even. So to complete the proof, we just need to show that with positive probability, C is ζ -separable and $|C| \leq \omega n$. (Note that if C is ζ -separable then it can be partitioned into sets from $\mathcal{B}_{\zeta, b}$.)

Let G be the graph where $V(G) = V(H)$ and for every $Z \in \binom{V(G)}{2}$, Z is an edge of G if $d_H(Z) \geq \zeta n$, i.e. the codegree of Z in H is at least ζn . We will now prove that with positive probability, the following two events occur:

- $|C| \leq \frac{1}{2}\omega n$,
- $\delta(G[C]) \geq \frac{\alpha\phi}{8f}n$.

First, the expected number of vertices added to W to form C is $\phi ft \leq \frac{\omega}{8}n$ plus potentially a few copies of F to make $b||C|$. By the second moment method, with probability at least $\frac{1}{4}$, at most $\frac{\omega}{4}n$ vertices are added to W so that $|C| \leq \frac{1}{2}\omega n$. Secondly, since $\delta(H) \geq \alpha n^2$ it is the case that $\delta(G) \geq \frac{\alpha}{2}n$. Indeed, if there was some vertex x with $d_G(x) < \frac{\alpha}{2}n$, then $d_H(x) \leq |N_G(x)| \cdot n + n \cdot \zeta n < (\frac{\alpha}{2} + \zeta)n^2$, a contradiction to the fact that $\delta(H) \geq \alpha n^2$ and $\zeta \leq \frac{\alpha}{2}$. Since $|W| \leq \frac{\omega\alpha}{4}n$, we have that any vertex x has at least $\frac{\alpha}{2}n - \frac{\omega\alpha}{4}n > \frac{\alpha}{4}n$ neighbors in G outside W . Since each F_i has size f , the vertex x therefore has a neighbor in G inside at least $\frac{\alpha}{4f}n$ of the copies of F . Therefore, the expected size of $\{y \in C : xy \in E(G)\}$ is at least $\frac{\alpha\phi}{4f}n$ and by Chernoff's Inequality (Lemma 15),

$$\mathbb{P} \left[|\{y \in C : xy \in E(G)\}| < \frac{\alpha\phi}{8f}n \right] \leq e^{-cn}$$

for some constant c . Thus n_0 can be selected large enough so that with probability at most $\frac{1}{4}$, there is some $x \in V(G)$ such that $|\{y \in C : xy \in E(G)\}| < \frac{\alpha\phi}{8f}n$. This implies that with probability at least $\frac{1}{2}$, $|C| \leq \omega n$ and $\delta(G[C]) \geq \frac{\alpha\phi}{8f}n$.

To complete the proof, we will show that $\delta(G[C]) \geq \frac{\alpha\phi}{8f}n$ implies that $G[C]$ has a perfect matching (which is equivalent to C being ζ -separable). Divide C into two equal sized parts C_1 and C_2 (recall that $|C|$ is even since b is even and $b||C|$). Such a partition exists since a random partition has this property with positive probability. Assume towards a contradiction that

Hall's Condition fails in $G[C_1, C_2]$, i.e. there exists a set $T \subseteq C_1$ such that $|N_G(T) \cap C_2| < |T|$. In a slight abuse of notation, let $\bar{T} = C_1 \setminus T$. Now $|T| \geq \frac{\alpha\phi}{16f}n$ since $\delta(G[C_1, C_2]) \geq \frac{\alpha\phi}{16f}n$. Similarly, $|\bar{T}| \geq \frac{\alpha\phi}{16f}n$ since if $z \in C_2 \setminus N_G(T)$ then $N_G(z) \cap C_1 \subseteq \bar{T}$. This implies that

$$|C_2 \setminus N_G(T)| = |C_2| - |N_G(T)| = |C_1| - |N_G(T)| > |C_1| - |T| = |\bar{T}| \geq \frac{\alpha\phi n}{16f}.$$

Since there are no edges of G between T and $C_2 \setminus N_G(T)$,

$$e_H(T, C_2 \setminus N_G(T), V(H)) \leq |T| \cdot |C_2 \setminus N_G(T)| \cdot \zeta n = \zeta |T| |C_2 \setminus N_G(T)| n. \quad (7)$$

On the other hand, since H is (p, μ) -dense,

$$e_H(T, C_2 \setminus N_G(T), V(H)) \geq p|T| |C_2 \setminus N_G(T)| n - \mu n^3.$$

Since $|T|$ and $|C_2 \setminus N_G(T)|$ are both larger than $\frac{\alpha\phi}{16f}n$,

$$e_H(T, C_2 \setminus N_G(T), V(H)) \geq \left(p - \mu \left(\frac{16f}{\alpha\phi} \right)^2 \right) |T| |C_2 \setminus N_G(T)| n$$

Since $\mu \leq \frac{p}{2} \left(\frac{\alpha\phi}{16f} \right)^2$, we have

$$e_H(T, C_2 \setminus N_G(T), V(H)) \geq \frac{p}{2} |T| |C_2 \setminus N_G(T)| n \quad (8)$$

Since $\zeta < \frac{p}{2}$, (8) contradicts (7). Therefore, $G[C_1, C_2]$ satisfies Hall's condition so that $G[C]$ has a perfect matching, i.e. C is ζ -separable. \square

Proof of Proposition 9. The proof is similar to the proof of Proposition 8 except Lemma 16 is used instead of Lemma 14. \square

3 Rich hypergraphs

This section contains the proofs of Theorems 3 and 4. By the previous section, these proofs come down to showing that (p, μ) -dense and large minimum degree imply either (a, b, ϵ, F) -rich or $(a, \mathcal{B}_{\zeta, b}, \epsilon, F)$ -rich, where we get to select a, b , and ϵ but $\zeta = \min\{\frac{p}{4}, \frac{\alpha}{4}\}$. As a warm-up before Theorem 3 (see Section 3.3), we start with the cherry.

3.1 Packing Cherries

Let $K_{1,1,2}$ be the cherry.

Lemma 17. *Let $0 < p, \alpha < 1$ and let $\zeta = \min\{\frac{p}{4}, \frac{\alpha}{4}\}$. There exists an $n_0, \epsilon > 0$, and $\mu > 0$ such that if H is an (n, p, μ, α) 3-graph with $n \geq n_0$, then H is $(4, \mathcal{B}_{\zeta, 4}, \epsilon, K_{1,1,2})$ -rich.*

Proof. Our main task is to come up with an $\epsilon > 0$ such that for large n and all $B \in \mathcal{B}_{\zeta,4}$, there are at least ϵn^4 vertex sets of size four which $K_{1,1,2}$ -absorb B ; we will define ϵ and μ later.

Fix $B = \{b_1, b_2, b_3, b_4\} \in \mathcal{B}_{\zeta,4}$, labeled so that $d_H(b_1, b_2) \geq \zeta n$ and $d_H(b_3, b_4) \geq \zeta n$. Let $X_1 = N(b_1, b_2) = \{x : xb_1b_2 \in E(H)\} \subseteq V(H)$ and $X_2 = N(b_3, b_4)$ and notice that $|X_1|, |X_2| \geq \zeta n$. Arbitrarily divide X_1 in half and call the two parts Y_1 and Y_2 . Let $\mu = \frac{p}{2}(\frac{\zeta}{2})^3$. Since $|Y_1|, |Y_2|, |X_2| \geq \frac{\zeta}{2}n = (2\mu/p)^{1/3}n$, by Lemma 13 there exists a $\xi > 0$ and n_0 such that $H[Y_1, Y_2, X_2]$ contains at least $\xi(\frac{\zeta n}{2})^4$ copies of $K_{1,1,2}$ with one degree two vertex in each of Y_1 and Y_2 and the degree one vertices in X_2 . The proof is now complete, since each of these cherries absorbs B . Indeed, let $\epsilon = \xi(\frac{\zeta}{2})^4$ and let $y_1 \in Y_1$, $y_2 \in Y_2$, and $x_1, x_2 \in X_2$ be such that $y_1y_2x_1, y_1y_2x_2 \in E(H)$. Then $A = \{y_1, y_2, x_1, x_2\}$ $K_{1,1,2}$ -absorbs B because $b_1b_2y_1, b_1b_2y_2 \in E(H)$ (recall that $Y_1, Y_2 \subseteq N(b_1, b_2)$) and similarly $b_3b_4x_1, b_3b_4x_2 \in E(H)$. Since there are at least ϵn^4 choices for y_1, y_2, x_1, x_2 , the proof is complete. \square

3.2 Packing Cycles

Throughout this section, let C_4 denote the hypergraph $C_4(2+1)$. This section completes the proof of Theorem 4.

Lemma 18. *Let $0 < p, \alpha < 1$ and let $\zeta = \min\{\frac{p}{4}, \frac{\alpha}{4}\}$. There exists an n_0 , $\epsilon > 0$, and $\mu > 0$ such that if H is a (n, p, μ, α) 3-graph with $n \geq n_0$, then H is $(18, \mathcal{B}_{\zeta,6}, \epsilon, C_4)$ -rich.*

Proof. Similar to the proof of Lemma 17, our task is to come up with an $\epsilon > 0$ such that for large n and all $B \in \mathcal{B}_{\zeta,6}$, there are at least ϵn^{18} vertex sets of size eighteen which C_4 -absorb B ; we will define ϵ and μ later.

Fix $B = \{b_1, b'_1, b_2, b'_2, b_3, b'_3\} \in \mathcal{B}_{\zeta,6}$ labeled so that $d_H(b_i, b'_i) \geq \zeta n$ for all i . For $1 \leq i \leq 3$, let $X_i = N(b_i, b'_i)$ and note that $|X_i| \geq \zeta n$. Now for each $1 \leq i \leq 3$, define

$$R_i = \left\{ \{r_1, r_2\} \in \binom{V(H)}{2} : |N(r_1, r_2) \cap X_i| \geq \frac{1}{10}p\zeta n \right\}.$$

In other words, R_i is the set of pairs with neighborhood in X_i at least one-tenth the “expected” size. If $|R_1| \leq \frac{1}{10}p\zeta n^2$, then

$$e(X_1, V(H), V(H)) \leq |R_1|n + \left(\binom{n}{2} - |R_1| \right) \frac{1}{10}p\zeta n \leq \frac{1}{5}p\zeta n^3. \quad (9)$$

On the other hand, since H is (p, μ) -dense,

$$e(X_1, V(H), V(H)) \geq p|X_1|n^2 - \mu n^3 \geq (p\zeta - \mu)n^3.$$

Let $\mu = \frac{p}{2}(\frac{1}{10}p\zeta)^3 < \frac{4}{5}p\zeta$ so that this contradicts (9). Thus $|R_1| \geq \frac{1}{10}p\zeta n^2$ and similarly for $1 \leq i \leq 3$, $|R_i| \geq \frac{1}{10}p\zeta n^2$.

Now fix $r_1r'_1 \in R_1$, $r_2r'_2 \in R_2$, and $r_3r'_3 \in R_3$. There are at least $(\frac{1}{10}p\zeta)^3 n^6$ such choices. For $1 \leq i \leq 3$ let $Y_i = N(r_i, r'_i) \cap X_i$ so $|Y_i| \geq \frac{1}{10}p\zeta n = (\frac{2\mu}{p})^{1/3}n$. By Lemma 13, there exists a

$\xi > 0$ such that there are at least $\xi(\frac{1}{10}p\zeta)^{12}n^{12}$ copies of $K_{4,4,4}$ across Y_1, Y_2, Y_3 . Let T_1, T_2, T_3 be the three parts of $K_{4,4,4}$ with $T_i \subseteq Y_i$ and let $T_i = \{y_1^i, y_2^i, y_3^i, y_4^i\}$.

Let $\epsilon = \xi(\frac{1}{10}p\zeta)^{15}$; we claim that there are at least ϵn^{18} vertex sets of size 18 which C_4 -absorb B . Indeed, $A := \{r_i, r'_i, y_j^i : 1 \leq i \leq 3, 1 \leq j \leq 4\}$ forms a C_4 -absorbing 18-set for B as follows. First, A has a perfect C_4 -packing: one C_4 uses vertices $r_1, r'_1, y_1^1, y_2^1, y_3^2, y_4^3$, another uses vertices $r_2, r'_2, y_1^2, y_2^2, y_3^3, y_4^1$, and the last uses $r_3, r'_3, y_1^3, y_2^3, y_4^4, y_1^4$. Secondly, $A \cup B$ has a perfect C_4 -packing: one C_4 using $b_1, b'_1, r_1, r'_1, y_1^1, y_2^1$, one using $b_2, b'_2, r_2, r'_2, y_1^2, y_2^2$, one using $b_3, b'_3, r_3, r'_3, y_1^3, y_2^3$, and one using $y_3^1, y_4^1, y_3^2, y_4^2, y_3^3, y_4^3$. Since there are $(\frac{1}{10}p\zeta)^3 n^6$ choices for $r_1, r'_1, r_2, r'_2, r_3, r'_3$ and then $\xi(\frac{1}{10}p\zeta)^{12}n^{12}$ choices for y_j^i , there are a total of at least ϵn^{18} choices for A . \square

Proof of Theorem 4. Apply Lemmas 17 and 18 and then Proposition 9. \square

3.3 Packing Linear Hypergraphs

In this section, we prove Theorem 3.

Lemma 19. *Let $0 < p, \alpha < 1$ and let F be a linear k -graph on f vertices. There exists an n_0 , $\epsilon > 0$, and $\mu > 0$ such that if H is a (n, p, μ, α) k -graph with $n \geq n_0$, then H is $(f^2 - f, f, \epsilon, F)$ -rich.*

Proof. Let $a = f(f - 1)$ and $b = f$. Similar to the proofs in the previous two sections, our task is to come up with an $\epsilon > 0$ such that for large n and all $B \in \binom{V(H)}{b}$, there are at least ϵn^a vertex sets of size a which F -absorb B ; we will define ϵ and μ later. Let $V(F) = \{w_0, \dots, w_{f-1}\}$ and form the following k -graph F' . Let

$$V(F') = \{x_{i,j} : 0 \leq i, j \leq f - 1\}.$$

(We think of the vertices of F' as arranged in a grid with i as the row and j as the column.) Form the edges of F' as follows: for each fixed $1 \leq i \leq f - 1$, let $\{x_{i,0}, \dots, x_{i,f-1}\}$ induce a copy of F where $x_{i,j}$ is mapped to $w_{i+j \pmod f}$. More precisely, if $\{w_{\ell_1}, \dots, w_{\ell_k}\} \in F$, then $\{x_{i,\ell_1-i \pmod f}, \dots, x_{i,\ell_k-i \pmod f}\} \in F'$. Similarly, for each fixed $0 \leq j \leq f - 1$, let $\{x_{0,j}, \dots, x_{f-1,j}\}$ induce a copy of F where $x_{i,j}$ is mapped to $w_{i+j \pmod f}$. Note that we therefore have a copy of F in each column and a copy of F in each row besides the zeroth row.

Now fix $B = \{b_0, \dots, b_{f-1}\} \subseteq V(H)$; we want to show that B is F -absorbed by many a -sets. Note that any labeled copy of F' in H which maps $x_{0,0} \rightarrow b_0, \dots, x_{0,f-1} \rightarrow b_{f-1}$ produces an F -absorbing set for B as follows. Let $Q : V(F') \rightarrow V(H)$ be an edge-preserving injection where $Q(b_j) = x_{0,j}$ (so Q is a labeled copy of F' in H where the set B is the zeroth row of F'). Let $A = \{Q(x_{i,j}) : 1 \leq i \leq f - 1, 0 \leq j \leq f - 1\}$ consist of all vertices in rows 1 through $f - 1$. Then A has a perfect F -packing consisting of the copies of F on the rows, and $A \cup B$ has a perfect F -packing consisting of the copies of F on the columns. Therefore, A F -absorbs B .

To complete the proof, we therefore just need to use Lemma 11 to show there are many copies of F' with B as the zeroth row. Apply Lemma 11 to F' where $m = f$, $s_1 =$

$x_{0,0}, \dots, s_f = x_{0,f-1}$ and $Z_{m+1} = \dots = Z_{f^2} = V(H)$. Since $\delta(H) \geq \alpha \binom{n}{k-1}$, (1) holds (with α replaced by $\frac{\alpha}{k^k}$) and since H is (p, μ) -dense (2) holds. Let $\gamma = \frac{1}{2} \left(\frac{\alpha}{k^k} \right)^{\sum d(x_{0,j})} p^{|F| - \sum d(x_{0,j})}$ and ensure that n_0 is large enough and μ is small enough apply Lemma 11 to show that

$$\text{inj}[F' \rightarrow H; x_{0,0} \rightarrow b_0, \dots, x_{0,f-1} \rightarrow b_{f-1}] \geq \gamma n^{f^2-f} = \gamma n^a.$$

Each labeled copy of F' produces a labeled F -absorbing set for B , so there are at least $\frac{\gamma}{a!} n^a$ F -absorbing sets for B . The proof is complete by letting $\epsilon = \frac{\gamma}{a!}$. \square

Proof of Theorem 3. Apply Lemma 19 and then Proposition 8. \square

4 Avoiding perfect F -packings

In this section we prove Theorem 5 using the following construction.

Construction. For $n \in \mathbb{N}$, define a probability distribution $H(n)$ on 3-uniform, n -vertex hypergraphs as follows. Let $G = G^{(2)}(n, \frac{1}{2})$ be the random graph on n vertices. Let X and Y be a partition of $V(G)$ where

- if $n \equiv 0 \pmod{4}$, then $|X| = \frac{n}{2} - 1$ and $|Y| = \frac{n}{2} + 1$,
- if $n \equiv 1 \pmod{4}$, then $|X| = \frac{n}{2} - \frac{1}{2}$ and $|Y| = \frac{n}{2} + \frac{1}{2}$,
- if $n \equiv 2 \pmod{4}$, then $|X| = |Y| = \frac{n}{2}$,
- if $n \equiv 3 \pmod{4}$, then $|X| = \frac{n}{2} - \frac{1}{2}$ and $|Y| = \frac{n}{2} + \frac{1}{2}$.

Let the vertex set of $H(n)$ be $V(G)$ and make a set $E \in \binom{V(G)}{3}$ into a hyperedge of $H(n)$ as follows. If $|E \cap X|$ is even, then make E into a hyperedge of $H(n)$ if $G[E]$ is a clique. If $|E \cap X|$ is odd, then make E into a hyperedge of $H(n)$ if E is an independent set in G .

Lemma 20. *For every $\epsilon > 0$, with probability going to 1 as n goes to infinity,*

$$\left| |E(H(n))| - \frac{1}{8} \binom{n}{3} \right| \leq \epsilon n^3.$$

Proof. For each $E \in \binom{V(H(n))}{3}$, E is a clique or independent set in $G(n, \frac{1}{2})$ with probability $\frac{1}{8}$. Thus the expected number of edges in $H(n)$ is $\frac{1}{8} \binom{n}{3}$ so the second moment method shows that with probability going to one as n goes to infinity, $|E(H(n)) - \frac{1}{8} \binom{n}{3}| \leq \epsilon n^3$. See [1] or the proof of Lemma 15 in [25] for details about the second moment method. \square

Lemma 21. *For every $\epsilon > 0$, with probability going to 1 as n goes to infinity the following holds. Let $X_1, X_2, X_3 \subseteq V(H(n))$. Then*

$$\left| e(X_1, X_2, X_3) - \frac{1}{8} |X_1| |X_2| |X_3| \right| < \epsilon n^3.$$

Proof. Let S_1, \dots, S_n be Steiner triple systems that partition $\binom{V(H(n))}{3}$. That is, view $V(H(n)) \cong \mathbb{Z}_{n-1}$ and for $1 \leq i \leq n$ let S_{i+1} consist of the triples $\{a, b, c\}$ such that $a+b+c = i \pmod{n}$. Each triple in $\binom{V(H(n))}{3}$ appears in exactly one S_i and two triples from the same S_i share at most one vertex.

Let $1 \leq i \leq n$ and let $X_1, X_2, X_3 \subseteq V(H(n))$. Let $e_H(X_1, X_2, X_3; S_i)$ be the number of ordered tuples $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$ such that $\{x_1, x_2, x_3\} \in E(H(n)) \cap S_i$. Let $e_{K_n}(X_1, X_2, X_3; S_i)$ be the number of ordered tuples $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$ such that $\{x_1, x_2, x_3\} \in S_i$.

The expected value of $e_H(X_1, X_2, X_3; S_i)$ is clearly $\frac{1}{8}e_{K_n}(X_1, X_2, X_3; S_i)$. If $E_1, E_2 \in S_i$ then since E_1 and E_2 share at most one vertex the events $E_1 \in E(H(n))$ and $E_2 \in E(H(n))$ are independent. By Chernoff's Bound (Lemma 15),

$$\mathbb{P} \left[\left| e_H(X_1, X_2, X_3; S_i) - \frac{1}{8}e_{K_n}(X_1, X_2, X_3; S_i) \right| > \epsilon |S_i| \right] < e^{-cn^2}.$$

for some constant c since $|S_i| = \frac{1}{n}\binom{n}{3}$ and the number of events is $e_{K_n}(X_1, X_2, X_3; S_i) < |S_i|$. By the union bound,

$$\mathbb{P} \left[\exists i, \exists X_1, X_2, X_3, \left| e_H(X_1, X_2, X_3; S_i) - \frac{1}{8}e_{K_n}(X_1, X_2, X_3; S_i) \right| > \epsilon |S_i| \right] < e^{-\frac{c}{2}n^2}.$$

Therefore, with high probability, for all i and all X_1, X_2, X_3 ,

$$\left| e_H(X_1, X_2, X_3; S_i) - \frac{1}{8}e_{K_n}(X_1, X_2, X_3; S_i) \right| < \epsilon |S_i|. \quad (10)$$

Summing (10) over i completes the proof. \square

Lemma 22. *Let F be a 3-graph with an even number of vertices such that there exists a partition of the vertices of F into pairs such that every pair has a common pair in their links. Then $H(n)$ does not have a perfect F -packing for any n .*

Proof. If $n \nmid v(F)$, then obviously $H(n)$ does not have a perfect F -packing. Therefore assume that $n \mid v(F)$ so that n is even. Let X and Y be the partition of $V(H(n))$ in the definition of $H(n)$. Since n is even, by definition both $|X|$ and $|Y|$ are odd. Let $\{w_1, z_1\}, \{w_2, z_2\}, \dots, \{w_{v(F)/2}, z_{v(F)/2}\}$ be the partition of $V(F)$ into pairs so that w_i and z_i have a common pair in their link for all i . By construction, if $x \in X$ and $y \in Y$ then there is no pair of vertices $u, v \in V(H(n))$ such that $xuv, yuv \in E(H(n))$ since the parities of $\{x, u, v\} \cap X$ and $\{y, u, v\} \cap X$ are different. This implies that for each i , w_i and z_i must either both appear in X or both appear in Y so that any copy of F in $H(n)$ uses an even number of vertices in X and an even number of vertices in Y . Since $|X|$ is odd, $H(n)$ does not have a perfect F -packing. \square

Proof of Theorem 5. By Lemmas 20, 21, and 22, with high probability $H(n)$ has the required properties. \square

5 Perfect Matchings in Sparse Hypergraphs

In this section, we prove Theorem 6. We follow the same outline as Section 3.

Lemma 23. *Let $k \geq 2$, $c > 0$, and a, b be multiples of k . There exists an n_0 depending only on k, a, b , and c such that the following holds for all $n \geq n_0$. Let H be an n -vertex k -graph, let $\mathcal{A} \subseteq \binom{V(H)}{a}$, and let $\mathcal{B} \subseteq \binom{V(H)}{b}$. Suppose that $\ell \geq cn^{a-1/2} \log n$ is an integer such that for every $B \in \mathcal{B}$ there are at least ℓ sets in \mathcal{A} which edge-absorb B . Then there exists set $A \subseteq V(H)$ such that A partitions into sets from \mathcal{A} and A edge-absorbs any set C satisfying the following conditions: $C \subseteq V(H) \setminus A$, $|C| \leq \frac{1}{64}\ell^2 n^{-2a+1}$, and C partitions into sets from \mathcal{B} .*

Proof. The proof is similar to Treglown-Zhao [33, Lemma 5.2] which in turn is similar to Rödl-Ruciński-Szemerédi [30, Fact 2.3]. Let $q = \frac{1}{8}\ell n^{-2a+1}$ and let $\mathfrak{A} \subseteq \mathcal{A}$ be the family obtained by selecting each element of \mathcal{A} with probability q independently. The expected number of intersecting pairs of elements from \mathfrak{A} is at most $q^2 \binom{n}{a} a \binom{n}{a-1} \leq \frac{1}{16}q\ell$. By Markov's inequality, with probability at least $\frac{1}{2}$ there are at most $\frac{1}{8}q\ell$ intersecting pairs of elements from \mathfrak{A} .

Now fix $B \in \mathcal{B}$ and let $\Gamma_B \subseteq \{A \in \mathcal{A} : A \text{ edge-absorbs } B\}$ be such that $|\Gamma_B| = \ell$. For each $A \in \Gamma_B$, let X_A be the event that $A \in \mathfrak{A}$. By Chernoff's Bound (Lemma 15),

$$\mathbb{P} \left[\left| |\Gamma_B \cap \mathfrak{A}| - q\ell \right| > \frac{1}{2}q\ell \right] \leq 2e^{-q\ell/6}.$$

Using that $\ell \geq cn^{a-1/2} \log n$, we have that $q\ell = \frac{1}{8}\ell^2 n^{-2a+1} \geq \frac{c^2}{8} \log^2 n$. By the union bound,

$$\mathbb{P} \left[\exists B, |\Gamma_B \cap \mathfrak{A}| < \frac{1}{2}q\ell \right] \leq \binom{n}{b} 2e^{-q\ell/6} \leq 2e^{b \log n - c^2 \log^2 n / 48} < \frac{1}{2}$$

for large n . Thus with probability at least $\frac{1}{2}$, \mathfrak{A} is such that for all $B \in \mathcal{B}$, there exist at least $\frac{1}{4}q\ell$ a -sets in \mathfrak{A} which edge-absorb B . Also, with probability at least $\frac{1}{2}$ there are at most $\frac{1}{8}q\ell$ intersecting pairs of elements from \mathfrak{A} .

Let \mathfrak{A}' be the subfamily of \mathfrak{A} consisting only of those a -sets A where A is not in any intersecting pair and also there is at least one $B \subseteq V(H)$ (of any size) such that A edge-absorbs B . Thus by the union bound, with positive probability \mathfrak{A}' is such that for all $B \in \mathcal{B}$, there exist at least $\frac{1}{8}q\ell$ a -sets in \mathfrak{A}' which edge-absorb B . Let \mathfrak{A}' be such a family of a -sets and let $A' = \cup \mathfrak{A}'$. First, $H[A']$ has a perfect matching. Indeed, each $A \in \mathfrak{A}'$ edge-absorbs some set so $H[A]$ has a perfect matching, and the sets in \mathfrak{A}' are disjoint so that these perfect matchings combine to form a perfect matching of $H[A']$. Second, A' partitions into sets from \mathcal{A} since the sets in $\mathfrak{A}' \subseteq \mathcal{A}$ are disjoint. Now let $C \subseteq V(H) \setminus A'$ with $|C| \leq \frac{1}{64}\ell^2 n^{-2a+1} = \frac{1}{8}q\ell$ and $C = B_1 \dot{\cup} \dots \dot{\cup} B_t$ with $B_i \in \mathcal{B}$. Using the bound on the size of C , we have that $t < \frac{1}{8}q\ell$. Since each B_i is edge-absorbed by at least $\frac{1}{8}q\ell$ sets in \mathfrak{A}' , each B_i can be edge-absorbed by a different a -set in \mathfrak{A}' . Therefore, $H[A' \cup B']$ has a perfect matching so the proof is complete. \square

Proof of Lemma 10. Let H' be the $v(F)$ -uniform hypergraph on the same vertex set as H , where $X \in \binom{V(H)}{v(F)}$ is a hyperedge of H' if $H[X]$ is a copy of F . Let $\ell = \lceil \epsilon n^a \rceil$ and notice since H is $(\mathcal{A}, \mathcal{B}, \epsilon, F)$ -rich, for every $B \in \mathcal{B}$ there are at least ℓ sets in \mathcal{A} which edge-absorb B in H' . Also, since a, b are multiples of $v(F)$ they are multiples of the uniformity of H' . Lastly, for large n we have that $\ell \geq n^{a-1/2} \log n$. Therefore, applying Lemma 23 (with $c = 1$) to H' shows that there exists a set $A \subseteq V(H') = V(H)$ such that A partitions into sets from \mathcal{A} and for any $C \subseteq V(H') \setminus A = V(H) \setminus A$ with $|C| \leq \frac{\epsilon^2}{64}n$ and C partitions into sets from \mathcal{B} , A edge-absorbs C in H' . Because each edge of H' is a copy of F , this implies that A F -absorbs C in H so the proof is complete by setting $\omega = \frac{\epsilon^2}{64}$. \square

Next, similar to the proofs in Sections 3.1, 3.2, and 3.3, we show that a bound on $\lambda_2(H)$ implies that each 3-set is edge-absorbed by many 6-sets. To do so, we need the hypergraph expander mixing lemma, first proved by Friedman and Wigderson [7, 8] (using a slightly different definition of $\lambda_2(H)$) and then extended to our definition of $\lambda_2(H)$ in [23].

Proposition 24. (*Hypergraph Expander Mixing Lemma [23, Theorem 4]*). *Let H be an n -vertex k -graph and let $S_1, \dots, S_k \subseteq V(H)$. Then*

$$\left| e(S_1, \dots, S_k) - \frac{k!|E(H)|}{n^k} \prod_{i=1}^k |S_i| \right| \leq \lambda_2(H) \sqrt{|S_1| \cdots |S_k|}.$$

Lemma 25. *Let $\alpha > 0$. Let H be a 3-graph and let $p = 6|E(H)|/n^3$. Assume $\delta_2(H) \geq \alpha p n$ and $\lambda_2(H) \leq \frac{1}{2}\alpha^2 p^{5/2} n^{3/2}$. Then for every $B \subseteq V(H)$ with $|B| = 3$, there are at least $\frac{1}{16}\alpha^4 p^5 n^6$ sets $A \subseteq V(H)$ with $|A| = 6$ such that A edge-absorbs B .*

Proof. Let $B = \{b_1, b_2, b_3\} \subseteq V(H)$. First, there are at least $\frac{1}{8}\alpha p n^3$ edges disjoint from B ; let $\{x_1, x_2, x_3\}$ be such an edge. For $1 \leq i \leq 3$, let $Y_i \subseteq N(b_i, x_i) = \{y : yx_i b_i \in H\}$ with $|Y_i| = \alpha p n$. Such a Y_i exists since the minimum codegree is at least $\alpha p n$. By the expander mixing lemma (Proposition 24),

$$e(Y_1, Y_2, Y_3) \geq p|Y_1||Y_2||Y_3| - \lambda_2(H) \sqrt{|Y_1||Y_2||Y_3|} \geq \alpha^3 p^4 n^3 - \lambda_2(H) \alpha^{3/2} p^{3/2} n^{3/2}.$$

Since $\lambda_2(H) \leq \frac{1}{2}\alpha^2 p^{5/2} n^{3/2}$,

$$e(Y_1, Y_2, Y_3) \geq \alpha^3 p^4 n^3 - \frac{1}{2}\alpha^{7/2} p^4 n^3 \geq \frac{1}{2}\alpha^3 p^4 n^3.$$

Let $\{y_1, y_2, y_3\}$ be an edge with $y_1 \in Y_1$, $y_2 \in Y_2$, and $y_3 \in Y_3$. Then $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ is a six-set that edge-absorbs B and there are at least $\frac{1}{8}\alpha p n^3 \cdot \frac{1}{2}\alpha^3 p^4 n^3 = \frac{1}{16}\alpha^4 p^5 n^6$ such sets. \square

Proof of Theorem 6. We are given $\alpha > 0$ such that $\delta_2(H) \geq \alpha p n$. Let $\gamma = 2^{-22}\alpha^{12}$.

First, we can assume that $p \geq \gamma n^{-1/10} \log^{1/5} n$. Indeed, by averaging there exists vertices s_1, s_2 such that the codegree of s_1 and s_2 is at most $2p n$. Then taking $S_1 = \{s_1\}$, $S_2 = \{s_2\}$, and S_3 as the non-coneighbors of s_1 and s_2 , Proposition 24 shows that

$$\lambda_2(H) \geq p\sqrt{|S_3|} \geq p\sqrt{(1-2p)n}.$$

But by assumption, $\lambda_2(H) \leq \gamma p^{16} n^{3/2}$. Therefore,

$$p\sqrt{(1-2p)n} \leq \gamma p^{16} n^{3/2}$$

which implies that $p \geq \gamma n^{-1/10} \log^{1/5} n$ (by a large margin).

By Lemma 25, for every $B \subseteq V(H)$ with $|B| = 3$ there are at least $\frac{1}{16}\alpha^4 p^5 n^6$ 6-sets $A \subseteq V(H)$ which edge-absorb B . Let $\ell = \frac{1}{16}\alpha^4 p^5 n^6$. If n is sufficiently large, then since $p \geq \gamma n^{-1/10} \log^{1/5} n$, we have that $\ell = \frac{1}{16}\alpha^4 p^5 n^6 \geq \frac{1}{16}\alpha^4 \gamma^5 n^{5.5} \log n$. Let $c = \frac{1}{16}\alpha^4 \gamma^5$ so that $\ell \geq cn^{5.5} \log n$. Now by Lemma 23, if n is sufficiently large there exists $A \subseteq V(H)$ such that A edge-absorbs all sets of size a multiple of three and at most

$$\frac{1}{64}\ell^2 n^{-11} = \frac{1}{2^{14}}\alpha^8 p^{10} n. \quad (11)$$

We now show how to construct a perfect matching in H . First, greedily construct a matching in $H[V(H) \setminus A]$. Say the greedy procedure halts with $B \subseteq V(H) \setminus A$ as the unmatched vertices. Since $3|v(H)|$ and $3|A|$ (since A is an edge-absorbing set), $3|B|$. By Proposition 24 (recall that $\gamma = 2^{-22}\alpha^{12}$),

$$e(B, B, B) = p|B|^3 \pm \lambda_2(H)|B|^{3/2} \geq p|B|^3 - \frac{1}{2^{22}}\alpha^{12} p^{16} n^{3/2} |B|^{3/2}. \quad (12)$$

If $|B| \geq 2^{-14}\alpha^8 p^{10} n$, then

$$p|B|^3 \geq p|B|^{3/2} \left(\frac{1}{2^{14}}\alpha^8 p^{10} n \right)^{3/2} = \frac{1}{2^{21}}\alpha^{12} p^{16} n^{3/2} |B|^{3/2}.$$

Combining this with (12) shows that $e(B, B, B) > 0$. This contradicts that the greedy procedure halted with B as the unmatched vertices. Thus $|B| \leq 2^{-14}\alpha^8 p^{10} n$ and then (11) shows that A edge-absorbs B , producing a perfect matching of H . \square

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